

# Digital Figures as Gateways to Linear Algebraic Objects: Supporting Observation and Conjecture Through Markov Chains

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**Abstract:** *Linear algebra is a conceptually rich and challenging subject, demanding that students reason flexibly about abstract mathematical objects such as matrices, vectors, and eigen-structures. To support this learning, we have developed digital interactive figures that allow students to explore linear algebraic relationships without the cognitive burden of routine computation. In this paper, we present a Markov chain assignment integrating two such digital figures alongside conceptual and proof-based problems, scaffolding students' exploration toward conjectures about the spectral properties of stochastic matrices and long-term behavior of the Markov chain. We discuss the pedagogical principles underlying our design, revisions following task-based student interviews, and implications for teaching linear algebra with technology-integrated materials.*

## 1. Introduction

Linear algebra is a foundational area of mathematics with wide-reaching applications across numerous disciplines, serving as a core component in fields such as data science, artificial intelligence, and various industrial technologies. Befitting these varied applied contexts, the role of technology in the application and utilization of linear algebra is clear, mirrored by the specific recommendations from the first and second Linear Algebra Curriculum Study Group (LACSG, LACSG 2.0; [1] and [9]) to increase curricular attention to abstract, formal topics and to improve technology integration in linear algebra education. Yet there is still a gap in studies of the deployment effects of technology usage in linear algebra classrooms [8].

As part of their work on the linear algebra text *Linear Algebra and its Applications* [3], the second author has developed a suite of digital algebraic modules programmed in Mathematica. These environments, here referred to as interactive figures, present students with non-coding computations to explore various computational and conceptual ideas in linear algebra. The primary aim of these collections of interactive figures is to provide students

with an accessible environment in which to encounter key ideas and patterns in linear algebra. In the past, we have presented these interactive figures as standalone experimental scenarios [4] meant to require only about 15 minutes of engagement and targeted specific and concise observations. Our most recent course module extends this pedagogy by scaffolding their exploration with conceptual questions and simple proof problems that stem from and set up further interactive figure inquiries.

In this paper, we will present a new suite of interactive figures focused upon properties of Markov chains especially as they apply concepts of eigentheory. While the topic of eigentheory in linear algebra education has received some attention ([10], [11]), educational studies involving Markov chains are light in the literature (e.g., [2]), possibly due to the topic being reserved for advanced or topics courses in linear algebra. The goal of this paper is to motivate a pedagogical inspiration for how and why these interactive figures have been developed, demonstrate our new model of interactive figure-integrated coursework, and to discuss the teaching implications of this new project.

## **2. The Objectives and Design of Interactive Figures**

### **2.1 Student Need for Familiarity with Linear Algebra Objects**

For many students, linear algebra serves as an introduction to mathematical objects beyond those encountered in traditional algebra and calculus. It is often the first context in which variables represent something other than scalar values or scalar-valued functions. The eigenvalue/eigenvector equation  $Ax = \lambda x$  captures this complexity:  $A$  is an  $n \times n$  matrix,  $x$  is a vector in  $\mathbb{R}^n$ , and  $\lambda$  is a scalar. To engage meaningfully with this equation and its attached concepts, students must recognize each variable as a distinct object and understand the two juxtaposed operations, matrix multiplication and scalar multiplication, yields comparable vectors.

When vectors are first introduced in linear algebra, students frequently draw upon geometric intuitions carried over from physics or vector calculus. While geometric thinking has genuine value in the subject, we believe over-reliance on geometry can hinder larger understandings, especially in higher dimensional systems that are opaque to geometric applications. With this in mind, the interactive figures to be demonstrated in this paper are designed to instead foreground the matrix as a rectangular array of numbers as the primary representation of matrices and vectors. This framing thus extends naturally to any finite dimension and is intended to support the development of computational skills.

## 2.2 Interactive Figure Design Pedagogy

The second author's early experience with creating MATLAB projects for introductory linear algebra students has been that they spent their energy wrestling with code and computation, and not on the linear algebraic objects. While computational training and coding fluency are certainly worthwhile pursuits in mathematics education, we intend the interactive figures herein to keep the coding in the background and give students quick and straightforward opportunities to view play with linear algebraic objects.

The design process of the interactive figures was shaped by two core student-centered principles: lowering the barrier to entry in terms of both time and computational complexity, and ensuring that students can interact meaningfully without needing prior technology or software expertise. On the instructional side, we prioritized automated example generation and built-in opportunities for students to engage in informal conjecturing based on their observations.

These design philosophies mean that students should be able to begin exploring with minimal setup or prerequisite knowledge. Operations such as matrix scaling, multiplication, and, in the present interactive figures, eigenvalue/eigenvector computation are performed by the worksheet itself. This is meant to allow students to focus on analyzing patterns and relationships rather than carrying out routine calculations. By removing these procedural burdens, the interactive figures seek to make it possible for students to quickly examine a wide variety of examples and more easily identify recurring behaviors or structures. As such, the functions available to students were limited to easy-to-use interfaces such as buttons and sliders that are immediately understandable to digital native students [7].

Via this design principle, we do not mean to dismiss the value of computational fluency and the practice so required; instead, these activities are meant to target conceptual fluencies and thus supplement standard computational coursework. As well, given the laborious and already strained for time nature of university mathematics learning, the streamlined nature of these interactive figures is thus meant to keep the time investment of the interactive figures reasonably low. It is therefore intentional and desired that students may spend relatively little time (on the order of 15 minutes) to gather some empirical data and develop notions of the concepts and relationships at play.

Inevitably, some students may wish to test their ideas beyond the worksheet's constraints, limited by the restricted input format. The goal is not open-ended experimentation, but structured interaction that promotes pattern recognition and supports the formulation of general principles. For students interested in deeper or alternative exploration, we subsequently refer

them to tools such as Wolfram Alpha and MATLAB.

### **2.3 Encouraging Mathematical Activity through Observation and Conjecture**

Our design builds upon a conceptual discovery cycle common in mathematical practice: Observation, Conjecture, Proof, and Theorem (OCPT). This cycle is in complement to the classical Definition-Lemma-Proof-Theorem-Proof-Corollary (DLPTPC) model of traditional mathematics. The DLPTPC model is a standard mode of communicating mathematical content, and thus a worthwhile fluency for mathematics students to acquire. Meanwhile, the OCPT model of discovery aims to engage students in the process of identifying and explicating the properties of mathematical phenomena, enabling them to gain greater ownership, insight, and understanding of the proof logic to follow and the conceptual connections between the objects being studied. It is also worth mentioning that the OCPT model is not meant to prescribe a perfectly linear process, but instead to invite a cyclic revision of ideas where, broadly, observations of mathematical phenomena lead to rules and statements about the structures or relationships involved (conjectures) which then demand some validation or rejection by proof to be accepted as fact.

As such, OCPT is appropriate for introducing mathematical content to students who can be invited into the mathematician's cycle of discovery and verification. A limitation of the OCPT model is that, by chance, examples may lead students to incorrect inferences, such as special cases where a non-general property does hold true. As such, the instructor's role as a curator capable of foreseeing and responding to such situations is an important safeguard.

The interactive figures we have designed for this assignment are thus intended to provide space for students to encounter mathematical relationships with which they are not already familiar, and to construct their own perspective on the structures at play. At the introductory linear algebra level, where many students have likely not encountered the techniques of formal mathematical proof, we intend to provide students with an arena to engage primarily in observation and conjecture. This work can be followed up on in class or subsequent assignments to develop the formalized proof justifying their conjectures. As such, each interactive figure was constructed with one or more conjectures that we would like students to pick up on, which we will discuss in Section 3. In our experience, students often miss patterns we intentionally designed the interactive figures to reveal, while also noticing unexpected features that were not part of our instructional focus. These occurrences further necessitate the role of the instructor as mediator, with the

interactive figures and assignments being actively attended to in other class discussions.

For instance, the below interactive figures are intended to allow students to contend with the properties of eigenvalues and eigenvectors as they relate to matrix addition, multiplication, and exponentiation. Each interactive figure is manipulable via a set of simple buttons, allowing students to vary the size of the matrices, scalar values (if applicable), and to generate new examples.

The top interactive figure (Figure 1) generates a matrix  $A$ , and with a set positive integer  $k$  displays  $A$ ,  $A^k$ , and the eigenvalues and corresponding eigenvectors of each. Behind the scenes, the interactive figure generates the  $A$ 's eigenvalues and eigenvectors first, selecting small-value integers, and then generates the corresponding matrix  $A$  via diagonalization. In this way, the eigenvalues can be plainly inspectable, even though the matrix's entries are typically large or unwieldy fractions.

As such, students may be able to readily observe that the eigenvalues of  $A^k$  appear to be the eigenvalues of  $A$  each raised to the  $k$  power, while the eigenvectors are exactly identical. In previous iterations of this interactive figure, the matrix was generated with small-value integer entries resulting in complicated decimal-valued eigenvalues and eigenvectors, which resulted in students being significantly less likely to make such observations about the properties of  $A^k$ .

The second interactive figure in Figure 1 below displays two randomly generated matrices,  $A$ , and  $B$ , as well as several arithmetic combinations of them via matrix addition and multiplication. As well, the eigenvalues of each is displayed. Students observing this interactive figure may as well notice several relationships among the eigenvalues, but there is also room left intentionally for students to expect a relationship to occur that actually does not. For instance, students might observe that  $AB$  and  $BA$  have the same eigenvalues, which is true, however they may also suspect that the eigenvalues of  $AB$  should be related to the eigenvalues of  $A$  and  $B$ , however no simple relationship is known. Similarly, after a few iterations it may be clear to students that the eigenvalues of  $A + B$  are unrelated to the eigenvalues of  $A$  and  $B$  separately.

It is entirely natural for students to expect some relationship to present itself while such a result is actually false, and we have found it valuable to allow for these non-relationships. Much like students must be taught that  $(a+b)^n \neq a^n + b^n$  in general, or that  $\det(A+B) \neq \det(A) + \det(B)$  generally, we mean to allow students to conjecture a result that is in fact false. As well, including these naively expected patterns might serve to normalize this feature of exploratory mathematics, a reality of practicing mathematicians,

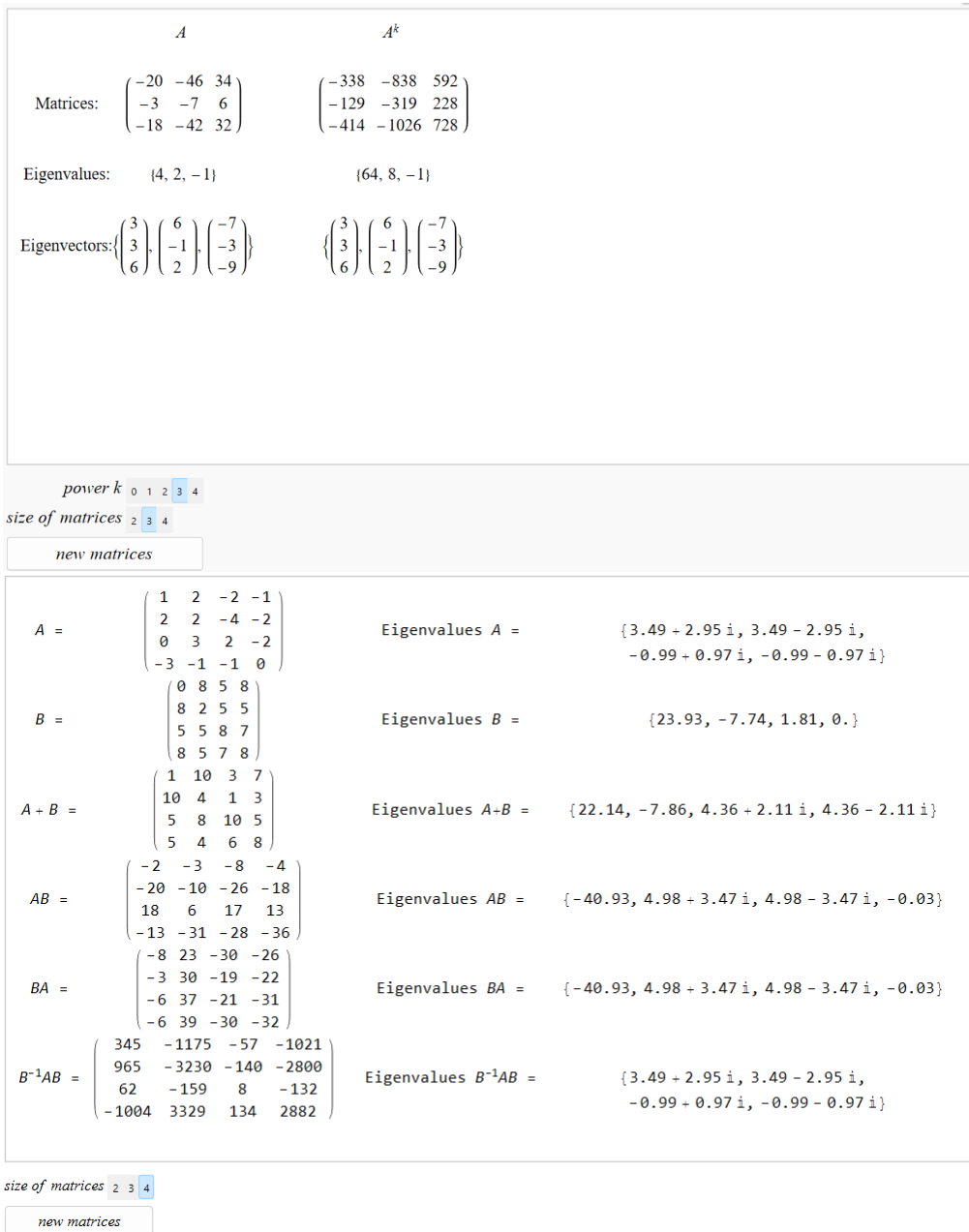


Figure 1: Interactive Figures Involving Matrix Exponentiation (top) and Matrix Addition and Multiplication (bottom)

where otherwise reasonable results are not guaranteed and should be duly scrutinized and verified.

## 2.4 Technology and Software Requirements

The interactive figures were developed in Mathematica using its ‘Manipulate’ function, allowing dynamic, self-contained modules that respond to sliders, buttons, and menus with no coding experience required. This streamlined design makes the tools accessible to students with varying technological proficiency. Mathematica’s support for combining text, visuals, and code enables the creation of narrative-style worksheets, with interactivity scaffolded by explanatory guidance.

While developing interactive figures requires a Mathematica license, students can access and use them freely via the Wolfram Player for Notebooks app distributed by Wolfram, ensuring broad usability regardless of institutional software access. The interactive figures themselves will be packaged with the digital content included in the linear algebra text *Linear Algebra and its Applications* [3].

### 3. The Markov Chain Assignment and Figures

In this section, we will show the two Markov chain interactive figures, as well as the associated assignment questions that prompt students to investigate these figures and apply their findings to solve problems.

#### 3.1 Introductory Markov Chain Content

The mathematics content to be delivered by this assignment can be decomposed into three inquiries: First, the transition matrix system  $x_{n+1} = Ax_n$  as it defines some iterative time process of interest; second, the properties of a stochastic transition matrix  $A$  (furthermore, this assignment provides regular stochastic matrices only, though the notion of regularity is not explored nor emphasized); and lastly, connecting the limiting behavior of the Markov chain to the probability eigenvector corresponding to eigenvalue 1 of the matrix  $A$ , namely that the Markov chain will converge to the eigenvalue-1 eigenvector scaled to a probability vector.

The first of the interactive figures we will discuss targets the properties of stochastic matrices. The activity introduces stochastic matrices with the following textbook [3] definition:

**Definition:** A stochastic matrix is an  $n \times n$  matrix whose columns are probability vectors, those being vectors whose entries are non-negative and sum to 1.

The resultant properties of stochastic matrices to explore include that powers of stochastic matrices are also stochastic (that is, the composition of two stochastic processes is a stochastic process), the value 1 always occurs

as an eigenvalue while the remaining eigenvalues are strictly less than 1 in magnitude, and eigenvectors corresponding to eigenvalues other than 1 must sum to 0. Since the eigenvalues of  $A^k$  are the eigenvalues of  $A$  raised to the  $k$  power,  $A^k$  will retain its eigenvalue-1 while its other eigenvalues will vanish as  $k$  grows to infinity. In this way, the long-term behavior of  $A^k$  becomes dominated by the eigenvector associated with eigenvalue-1.

### 3.2 Assignment Introduction: Partitioned Population Dynamics

This assignment seeks to introduce fundamental Markov chain ideas through a contextualized problem scenario. One of the most readily accessible real-world contexts utilizing Markov chains is population dynamics in a partitioned population. Thus, the problem statement was adapted from [3] to prompt a two-population Markov relation, which students are instructed to create the following annual transition matrix and initial state vector:

$$A = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}, x_0 = \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix}$$

Once the matrix in question is identified, students are asked to consider how the population changes across several years, at 2 years, 5 years, and a general  $k$  years. This problem is intended to get students to think about the iterative nature of Markov chains and potentially its connection to matrix exponentiation, which will be called upon in the next section.

### 3.3 Interactive Figure 1: Investigating the Properties of Stochastic Matrices

Consider the two images below of the first interactive figure focused on the eigenvalue properties of stochastic matrices (Figure 2). The interactive figure generates a matrix  $A$ , then computes  $A^k$  for the selected scalar  $k$ , as well as the eigenvalues and eigenvectors of both  $A$  and  $A^k$ . The displayed eigenvalues are given in descending order of magnitude, while the eigenvectors are listed vertically associated with their corresponding eigenvalue.

As standard for our matrix interactive figures, the student is able to select the size of the matrix (up to window display constraints) and the scalar exponent  $k$  via button controls. Two “new matrix” buttons then allow the student to decide to generate new examples of either stochastic or non-stochastic matrices. Modifying the size of the matrix regenerates a random matrix of the previously specified type (stochastic or non-stochastic), while changing the exponent  $k$  does not change the base matrix  $A$ . Figure 2 (top) depicts a typical result of generating a new stochastic matrix, with dimensions set to

$3 \times 3$  and exponent set to 2. Figure 2 (bottom) then depicts a typical result of generating a non-stochastic matrix, this time with dimensions  $4 \times 4$  and exponent 3.

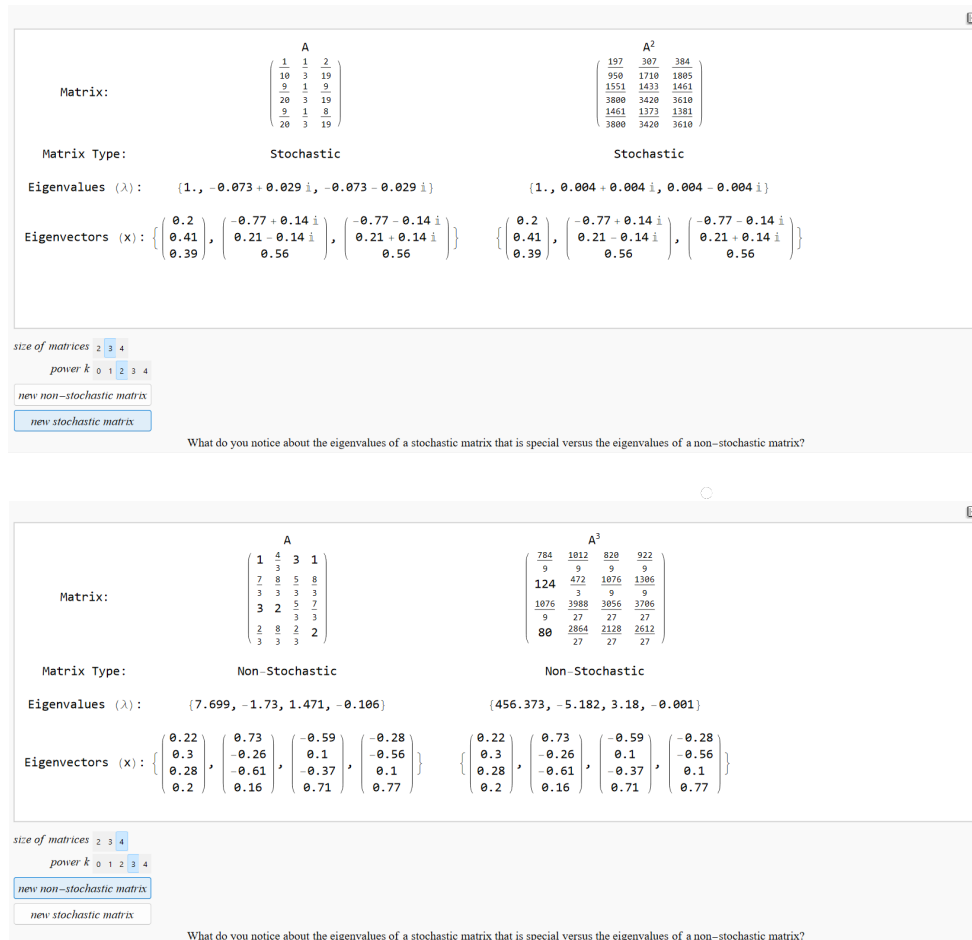


Figure 2: Powers of Stochastic (top) and Non-Stochastic (bottom) Matrices Interactive Figure

At the time this assignment would be given, students should be familiar with basic algebraic properties of eigenvalues and eigenvectors, for instance, that the eigenvalues of  $A^k$  are the eigenvalues of  $A$  individually raised to the  $k$  power, or that the eigenvectors of  $A^k$  are the same for all positive integer values of  $k$ . Nevertheless, these properties are observable in this interactive figure.

However, the intended focus of this interactive figure is on the properties of stochastic matrices and their powers. Students are prompted to experi-

ment with this interactive figure in Question 3 of the assignment, which asks them to note anything they "notice about the eigenvalues and eigenvectors of stochastic matrices compared to non-stochastic matrices?"

Through example generation, students could notice that 1 always occurs as an eigenvalue of a stochastic matrix, and furthermore that this eigenvalue is the largest in magnitude. However, this observation becomes trickier for novice students when imaginary eigenvalues occur. This potential observation is in contrast to the non-stochastic matrices, which are capable of generating leading eigenvalues greater or less than 1.

A deeper observation, utilizing the notion that matrix exponentiation applies to the eigenvalues of the original matrix, is that, since all eigenvalues of a stochastic matrix besides eigenvalue 1 are then strictly less than one, larger exponents will cause the tailing eigenvalues to shrink. Even as the interactive figures currently bound the largest exponent to 4, within the necessary rounding of the figure's eigenvalue display, it can be seen that the tailing eigenvalues of stochastic  $A^k$  are diminishing. While possible that these eigenvalues could be quite close to 1, and thus shrink only very slowly upon exponentiation, in practice this is a rare occurrence for the randomization of the interactive figure's generation method. Such an observation can later be connected to the notion that the Markov chain iteration of a regular stochastic matrix will become dominated by the eigenvector corresponding to the eigenvalue 1, as the remaining eigenvalues tend to 0 for large exponents  $k$ .

Furthermore, given the probability vector definition of stochastic matrices provided, students may be inclined to investigate further the eigenvectors of stochastic matrices. As such, it can be observed that the eigenvector corresponding to eigenvalue 1 of a stochastic matrix is a probability vector, whereas the remaining eigenvectors' entries must sum to 0. While the nature of the probability eigenvector of eigenvalue 1 was an intended observation of this interactive figure, we were not aware of the sum-to-0 property of the remaining eigenvectors until this feature was pointed out to us by a student using the interactive figures. While far beyond the ability of introductory students to analyze, this is a true result. The proof justifying this relies upon the orthogonality of left and right eigenvectors, a concept not explored at the introductory level. As with observations about the eigenvalues of stochastic matrices, the non-stochastic matrices defy these patterns and serve to identify stochastic matrices as specially possessing these properties.

### **3.4 Subsequent Conceptual Questions**

Following the above directed observational work with the first interactive figure, the students are tasked with a few introductory-accessible conceptual

and simple algebraic proof problems. These problems are meant to reinforce the conjectures students may have made earlier (or point them towards these ideas, if not), as well as prime students for future observations in the final interactive figure.

- 4a. Verify if  $\lambda = 1$  is an eigenvalue of  $A$ , then 1 is an eigenvalue of  $A^k$ .
- 4b. What do you notice about the size of the other eigenvalues of  $A^k$ ? Explain why you expect this to happen.
- 5a. Verify  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  is an eigenvector of  $A$  (the matrix you constructed in Question 1) corresponding to eigenvalue 1.
- 5b. Find an eigenvector of your matrix  $A$  from (1.) corresponding to eigenvalue 1 that is also a probability vector.

The problems shown above are intended to extend from the observations students made in working with the interactive figures. For instance, after observing that 1 is an eigenvalue of a stochastic matrix (and perhaps as well an eigenvalue of  $A^k$ ), Question 4.a. above codifies this relationship via an introductory-level proof problem requiring only basic manipulation of the fundamental eigenvalue-eigenvector equation.

Question 5 then seeks to prepare students for the main observation of the final interactive figure, that being that the steady-state vector of a Markov chain is the eigenvector of the matrix corresponding to eigenvalue 1, expressed as a probability vector. Providing the eigenvalues of the matrix and asking students to confirm a given eigenvector follows our principle of minimizing computation in favor of conceptual emphasis, while still encouraging students to bring to mind and engage with the fundamental equation  $Ax = \lambda x$ . Furthermore, scaling the eigenvector to be a probability vector prepares students to recognize this vector as the Markov chain limit, rather than the integer-valued eigenvector provided in part (5a) that would likely first arise from standard row reduction.

### **3.5 Culminating Interactive Figure and Observations**

The final section of this assignment seeks to unite the observed properties of stochastic matrices with the long-term behavior of Markov chains. The question prompt asks students to consider the second interactive figure (Figure 3 below), and to adjust the initial population vector and observe what happens when the Markov chain is iterated, and how this behavior changes when the initial population is varied.

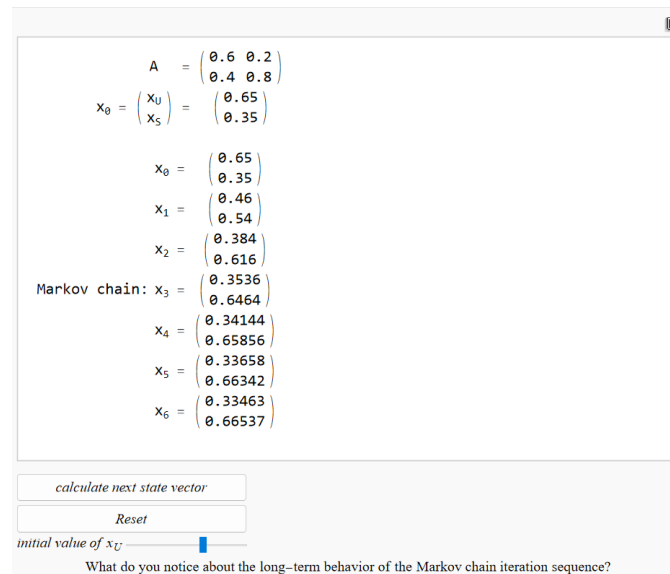


Figure 3: Markov Chain Iteration Interactive Figure

In the interactive figure, the transition matrix defined in by the transition system is provided, along with an adjustable  $x_0$  probability vector. Users may apply  $A$  iteratively, computing  $x_n = Ax_{n-1}$ , with the ability to reset the chain to test new initial vectors. Two main observations are intended to be made, directed by the question prompt: first, the sequence of iterates approaches the eigenvector calculated in question (5.b); and second, that this convergence occurs for any chosen vector  $x_0$ .

While the ideas in this interactive figure are more complex than we could reasonably expect students to work through algebraically, the observations made in the figures still provide useful introductory leverage for students first engaging with Markov chains. Furthermore, as discussed in the pedagogy of the Observation and Conjecturing levels of activity, these observations can then be followed up upon in subsequent class time, where an instructor can demonstrate further or more complex ideas.

### 3.6 Data-Based Revisions to the Markov Chain Assignment

The state of the Markov chain assignment as presented is adapted from the interview task for which we collected student data reported in [5]. In this early version of the lesson, students were given more direct algebraic tasks such as calculating the eigenvalues of the  $2 \times 2$  population matrix and the eigenvector corresponding to eigenvalue 1. After the interview trials, those computational problems that scaffolded the interactive figures were replaced with the conceptual and proof-like problems discussed here.

These changes were made in response to some of the difficulties experienced and shortcomings of the assignment observed during these interviews. For instance, when the participants calculated the eigenvectors of the stochastic Markov chain matrix, it did not occur to them nor were they directed to consider the probability matrix in that eigenspace. As such, when the participants proceeded to interact with the second interactive figure, they were readily able to notice that the Markov chain would converge to the same vector limit regardless of initial state, however they were not able to recognize this limit vector as they eigenvector they had just computed. As such, the current version of this assignment provides a (non-probability) eigenvector of the Markov matrix corresponding to eigenvalue 1, and then asks students to scale this vector in order to be a probability vector. In our live class implementations of this activity, this amendment affords the students to much more readily connect the eigenvector with the Markov limit.

#### **4. Concluding Remarks and Implications for Teaching**

This work represents our ongoing effort to refine the role of digital interactive figures in fostering conceptual and theoretical understanding in linear algebra, using observation and conjecture as a framework for authentic mathematical engagement. As instructors, we are still actively working through questions of balance and how much freedom to give students in their exploration versus how much to guide them toward specific conceptual targets, as well as how to situate these activities within applied contexts that are both meaningful and illuminating. The use of an application such as a population change Markov chain demonstrated a productive avenue of making abstract ideas more tangible and interpretable, though we will continue to consider how such contexts can most effectively shape and motivate student reasoning.

As linear algebra instructors, we have found great value in these interactive figures and their matrix-vector visualizations in familiarizing and demystifying these objects for students. Due to the symbolically dense nature of linear algebraic notation, it is our experience that students struggle with differentiating the identities and relevant structures of these objects. Thus, before even approaching the targeted conceptual understandings, these interactive figures have been greatly productive for us in enabling students to materially view and internalize the different objects at play.

It is worth reiterating that, by design, these interactive figure assignments place computational load upon the digital tool and so are not a substitute for traditional coursework and practice. Instead, they are intended to supplement such coursework and have thus served best as a tool for introducing

new content pre-lecture before students have the computational or algebraic tools available themselves. These nascent student observations can then be integrated into course delivery, with those ideas verified and built upon by the instructor.

These interactive figures and the Markov chain assignment discussed herein in particular will be available as part of the digital support provided by the *Linear Algebra and its Applications* text [3]. In the first and second author's department, these interactive figures are being deployed across all sections of introductory linear algebra for review and feedback. Our design process is, by nature, iterative. Each time these materials are implemented, we gain new understanding of how students read, navigate, and reason about the mathematical structures we present. We see this work as part of a larger pedagogical inquiry into how technology can serve not only as a tool for visualization or computation, but as a medium for genuine mathematical activity.

Looking ahead in the short term, we are continuing to analyze data from task-based interviews where students participated in this Markov chain assignment, in order to substantiate our interactive figure-integrated Observation Conjecture Proof Theorem model of doing mathematics [6]. We are also exploring potential extensions of this assignment, including a precursor activity that introduces the power method and its connection to Markov chain convergence. In the broader future, we hope the linear algebra community will utilize and assist us in refining and expanding this system of interactive figures-based instruction. In the future, after this qualitative analysis, we hope to perform quantitative analyses of student performance mediated by the interactive figures to further assess the impact and merits of the interactive figures.

## Supplementary Electronic Materials

A Mathematica file collecting the interactive figures displayed and discussed in Figures 1, 2, and 3:

Dropbox Download Mathematica File

## References

- [1] Carlson, D., Johnson, C., Lay, D., & Porter, A. D. (1993). The Linear Algebra Curriculum Study Group recommendations for the first course in linear algebra. *College Mathematics Journal*, 24, 41-46.
- [2] Harsy, A., & Smith, M. (2024). An Application Approach to Teaching Linear Algebra. *PRIMUS*, 1-20. <https://doi.org/10.1080/10511970.2024.2361376>

- [3] Lay, D. C., Lay, S. R., & McDonald, J. J. (2021). *Linear Algebra and Its Applications*. Sixth edition. Pearson.
- [4] Peffer, R., McDonald, J., & Stewart, S. (2023). Incorporating digital interactive figures: Facilitating student exploration into properties of eigenvalues and eigenvectors. In *Proceedings of the 28th Asian Technology Conference in Mathematics (ATCM)* (pp. 342-351).
- [5] Peffer, R., McDonald, J., & Stewart, S. (in press). Analyzing a Student's Observations and Conjectures about Markov Chains: Identifying Scheme Elements and their Instrumentation of Digital Figures. In *Proceedings of the 28th annual Conference on research in Undergraduate Mathematics education*.
- [6] Peffer, R., McDonald, J., & Stewart, S. (2026). Analyzing a Student's Observations and Conjectures About Markov Chains: Identifying Scheme Elements and Their Instrumentation of Digital Figures. *Proceedings of the 28th Annual Conference for Research in Undergraduate Mathematics Education*.
- [7] Prensky, M., & Berry, B. D. (2001). Do they really think differently. *On the horizon*, 9(6), 1-9.
- [8] Stewart, S. Andrews-Larson, C., & Zandieh, M. (2019). Linear algebra teaching and learning: Themes from recent research and evolving research priorities, *ZDM Mathematics Education*, 51(7), 1017-1030.
- [9] Stewart, S., Axler, S., Beezer, R., Boman, E., Catral, M., Harel, G., McDonald, J., Strong, D., & Wawro, M. (2022). The linear algebra curriculum study group (LACSG 2.0) recommendations. *The Notices of American Mathematical Society*, 69(5), 813-819.
- [10] Thomas, M., & Stewart, S. (2011). Eigenvalues and eigenvectors: Embodied, symbolic, and formal thinking. *Mathematics Education Research Journal*, 23, 275–296.
- [11] Wawro, M., Watson, K., & Zandieh, M. (2019). Student understanding of linear combinations of eigenvectors. *ZDM Mathematics Education*, 51 (7), 1111-1124.